Prop 9.2.15

Assume that the Semi-Cartan graph is connected and satisfies (CG3)

Let $X \in \mathcal{X}$, TFAE:

(1) $\exists\, m \in \mathbb{N}$, s.t $\forall Y \in \mathcal{X}$ any ~~X~~ Y-reduced sequence has length at most m

(2) $\exists\, m \in \mathbb{N}$, s.t any X-reduced ---

(3) $\mathcal{G}$ is finite.

Proof: $(1) \Rightarrow (2)$

$(2) \Rightarrow (3)$:

It suffices to prove $|\Delta^{X\,re}| < \infty$.

If not, $|\Delta^{X\,re}| = \infty$

Let $(i_1, \cdots, i_\ell)$ be an X-reduced sequence of maximal length

Let $Y = r_{i_\ell} \cdots r_{i_1}(X)$

Then $(i_\ell, \cdots, i_1)$ is Y-reduced

Moreover, $|\Delta_+^{X\,re}| > \ell$

by $\boxed{(\mathrm{CG3})}$ and $\underline{\text{Lem 9.2.8}}$

$\exists\, i$ s.t $(i, i_\ell, \cdots, i_1)$ is $r_i(Y)$-reduced

so $(i_1, \cdots, \underline{i_\ell}, i)$ is X-reduced

↯ to ~~[illegible]~~ maximal length

$(3) \Rightarrow (1)$

Since $\mathcal{G}$ is connected,

$\underline{\text{Lem 9.1.18}}$ and $(3) \Rightarrow \boxed{\bigcup_{Y \in \mathcal{X}} \Delta^{Y\,re}}$ is a finite set.

Assume $\mathcal{G}$ is finite, $\mathcal{Y} = \{r_{i_1} \cdots r_{i_n}(X)\}$

Then $\bigcup_{Y \in \mathcal{Y}} \Delta^{Y\,re}$ is a finite set

Let $\boxed{m = \left|\bigcup_{Y \in \mathcal{X}} \Delta^{Y\,re}\right|}$

Then $\forall Y \in \mathcal{X}$, $\forall$ Y-reduced sequence has length at most m.

$\Lambda^{Y,K} = \{\underline{\beta_1, \cdots \beta_\ell}\}$

$\mathcal{G}$ connected finite

$\Rightarrow \forall X \in \mathcal{X}$, $|\Delta^{X\,re}| = \boxed{m}$

X-reduced K

$\Lambda^{X,K}$

§9.3 Weak exchange condition and longest elements

Lem 9.3.1

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph,

$x \in \mathcal{X}$, $i \neq j$. TFAE:

(1) $a_{ij}^{x} = a_{ji}^{x} = 0$

(2) $m_{ij}^{x} = 2$

(3) $\Delta^{x\,re} \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j) = \{\alpha_i, \alpha_j\}$

Proof:

$(2) \Leftrightarrow (3)$ obvious

$① \Rightarrow ②$ Assume (1)

then $0 = a_{ij}^{x} = a_{ij}^{r_i(x)}$

$\Rightarrow a_{ji}^{r_i(x)} = 0$

$\Rightarrow a_{ji}^{r_j r_i(x)} = 0$

Hence $K_{ij}^{x} = \underline{(i, j)}$ then $\overline{m}_{ij}^{x} = 2$

$i, j, i, j \cdots$

$\parallel$ Cartan graph

m_{ij}^{x}

$\underline{(i, j, i)}$

$s_i^{r_i(x)} s_j^{r_j r_i(x)} (\alpha_i)$

$= s_i^{r_i(x)} (\alpha_i) = -\alpha_i$

$② \Rightarrow ①$

$(2) \Rightarrow s_i^{r_i(x)}(\alpha_j) = \alpha_j$

$\Rightarrow a_{ij}^{r_i(x)} = 0 = a_{ji}^{r_i(x)}$

$\Rightarrow a_{ij}^{x} = 0 = a_{ji}^{x}$ □

Rmk 9.3.2

In semi-Cartan $(2) \Leftrightarrow (3)$

$(2) \Rightarrow (1)$

$(1) \not\Rightarrow (2)$

$\overline{m}_{ij}^{x} \overset{?}{=} m_{ij}^{x}$

Lemma 9.3.3

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ Cartan graph,

$i, j \in I$, $a_{ij}^{x} = 0$

Then $\underline{a_{jl}^{x}} = \underline{a_{jl}^{r_i(x)}}$, $\forall l \in I$.

Proof: $a_{ij}^{x} = 0 \Rightarrow a_{ij}^{r_i(x)} = 0 \Rightarrow a_{ji}^{r_i(x)} = 0$

We have $\boxed{m_{ij}^{x} = 2}$ by Lem 9.3.1

Now by the coxter relations

$\boxed{\mathrm{id}_x s_i s_j = \mathrm{id}_x s_j s_i}$

$\Rightarrow \underline{s_i s_j^{x}(\alpha_l) = s_j s_i^{x}(\alpha_l)}$

$\Rightarrow$ — — ....

$\Rightarrow a_{jl}^{x} = a_{jl}^{r_i(x)}$ □

Thm 9.3.4

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a Cartan graph,

$X \in \mathcal{X}$, $k \in \mathbb{N}$, $i_1, \dots, i_k, i \in I$.

Assume that $\mathrm{id}_X \underline{s_{i_1} \cdots s_{i_k}} (\alpha_i) \in -\mathbb{N}_0^I$

Then $\exists\ j_1, \dots, j_k \in I$, s.t $j_k = i$

$\underline{\mathrm{id}_X s_{i_1} \cdots s_{i_k}} = \underline{\mathrm{id}_X s_{j_1} \cdots s_{j_k}}$

Proof: Let $w = \mathrm{id}_X s_{i_1} \cdots s_{i_k}$ $K = (i_1, \dots, i_k)$

If K is $\underline{\text{reduced}}$, then by prop 9.2.16

Assume that K is not reduced,

$\exists\ 1 \le \ell \le k$, s.t $\beta_\ell^{X,K} \in -\mathbb{N}_0^I$

$\beta_n^{X,K} \in \mathbb{N}_0^I \quad 1 \le n < \ell$

$(i_1, \dots, i_{\ell-1})$ is X-reduced

Hence by prop 9.2.16

$\exists j_1, \dots, j_{\ell-1} \in I \quad j_{\ell-1} = i_{\ell-1}$

s.t $\mathrm{id}_X s_{i_1} \cdots s_{i_{\ell-1}} = \mathrm{id}_X s_{j_1} \cdots s_{j_{\ell-1}}$

Then $\mathrm{id}_X \underline{s_{i_1} \cdots s_{i_{\ell-1}}} \boxed{s_{i_\ell}} \cdots s_{i_k}$

$= \mathrm{id}_X s_{j_1} \cdots s_{j_{\ell-2}} s_{i_{\ell+1}} \cdots s_{i_k} s_i^2$

□

Thm 9.3.5

Let $\mathcal{G} = \mathcal{G} = (I, \mathcal{X}, r, A)$

$w = \mathrm{id}_X s_{i_1} \cdots s_{i_\ell} \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$

$\beta_k = \mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}} (\alpha_{i_k}) \quad 1 \le k \le \ell$

(1) TFAE: $w = \mathrm{id}_X s_{i_1} \cdots s_{i_k}$

(a) $\underline{(i_1, \dots, i_\ell)}$ is a reduced decomposition of w

(b) $(i_1, \dots, i_\ell)$ is X-reduced

(2) $N(w) = \ell(w)$.

if $(i_1, \dots, i_\ell)$ is a reduced de— of w, then $\Delta^{X\,re}(w) = \{\beta_1, \dots, \beta_\ell\}$

$\overset{\shortparallel}{\Lambda^X(i_1, \dots, i_\ell)}$

Proof:

Let $K = (i_1, \dots, i_\ell)$

Suppose (1)(a), and $(i_1, \dots, i_\ell)$ is not reduced

then $\exists\ 2 \le k \le \ell$ s.t $\beta_k \in -\mathbb{N}_0^I$,

by Thm 9.3.4 $\exists j_1, \dots, j_{k-1}$, $j_{k-1} = i_k$

s.t $\mathrm{id}_X s_{i_1} \cdots s_{i_{k-1}} = \mathrm{id}_X s_{j_1} \cdots s_{j_{k-1}}$

Then $\mathrm{id}_X s_{i_1} \cdots s_{i_\ell} = \underline{\mathrm{id}_X s_{j_1} \cdots s_{j_{k-2}} s_{i_{k+1}} \cdots s_{i_\ell}}$

$\nleftrightarrow$ to reduced decomposition

Hence $(i_1, \dots, i_\ell)$ is X-reduced.

$\underline{\Lambda^X(K)} \overset{定义}{=} \{\beta_1, \dots, \boxed{\beta_\ell}\} = \left|\Delta^{X\,re}(w)\right|$

$\Rightarrow N(w) = \ell(w)$

Suppose (1) (b)

By previous paragraph,

$N(w) = \ell$

$\Rightarrow \ell(w) = N(w) = \ell$

$\Rightarrow \mathrm{id}_X s_{i_1} \cdots s_{i_\ell}$ is a reduced decomp of w

□

Cor 9.3.6

Let $\mathcal{G}$ Cartan graph, $w \in W(\mathcal{G})$

(1) $w(\alpha_i) \in \mathbb{N}_0^I \Leftrightarrow \ell(ws_i) = \ell(w)+1$

(2) $\underline{w(\alpha_i) \in -\mathbb{N}_0^I} \Leftrightarrow \ell(ws_i) = \ell(w)-1$

Proof: 两个 "⇒" 由 9.1.21 (2) since $N(w) = \ell(w)$

两个 "⇐" since $\underline{w(\alpha_i) \in \mathbb{N}_0^I \cup -\mathbb{N}_0^I}$

□

Cor 9.3.7

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ Cartan

w, If $\alpha_i \in \underline{\Delta^{Xre}(w)}$

then ∃ a reduced decomp $(i_1, \ldots, i_\ell)$ with $i_1 = i$

Proof:

Assume $\alpha_i \in \Delta^{Xre}(w)$,

then $w^{-1}(\alpha_i) \in -\mathbb{N}_0^I$

hence $\ell(\underline{w^{-1}s_i}) = \ell(w^{-1}) - 1$

Let $\ell = \ell(w) = \ell(w^{-1})$

$w' = s_i w$

then $\ell(w') = \ell - 1$

~~Suppose~~ Let w' 的一个 reduced 分解为 $(i_2, \ldots, i_\ell)$

then $w = s_i w'$

then $(i, i_2, \ldots, i_\ell)$ is a reduced dec of w. □

Cor 9.3.8

$\mathcal{G}$ Cartan graph, X, $w, w' \in \mathrm{Hom}(W(\mathcal{G}), X)$

(1) Assume $w(\alpha_i) \in \mathbb{N}_0^I$, $\forall i$, then $w = \mathrm{id}_X$

Proof: we have $N(w^{-1}) = 0 = \ell(w^{-1})$

$$\Rightarrow w = \mathrm{id}_X \quad \overset{\shortparallel}{\ell(w)}$$

(2) Assume that $\Delta^{X\,\mathrm{re}}(w) = \Delta^{X\,\mathrm{re}}(w')$

then $w = w'$

Proof:

Let $i \in I$,

we will prove $w^{-1}w'(\alpha_i) \in \mathbb{N}_0^I$ $\forall i$

then by (1) ...

If $w'(\alpha_i) \in \mathbb{N}_0^I$,

then $w'(\alpha_i) \in \Delta^{X\,\mathrm{re}}(w') = \Delta^{X\,\mathrm{re}}(w)$

then $w^{-1}w'(\alpha_i) \in \mathbb{N}_0^I$

If $w'(\alpha_i) \in -\mathbb{N}_0^I$

then $-w'(\alpha_i) \in \Delta^{X\,\mathrm{re}}(w')$

$= \Delta^{X\,\mathrm{re}}(w)$

then $-w^{-1}w'(\alpha_i) \in -\mathbb{N}_0^I$

then $\underbrace{w^{-1}w}_{=\mathrm{id}}(\alpha_i) \in \mathbb{N}_0^I$

(3). Assume that $\Gamma(w) = \Gamma(w')$

then $w = w'$.

Proof: by (2).

$$\Downarrow$$

$$\Delta^{X\,\mathrm{re}}(w) = \Delta^{X\,\mathrm{re}}(w')$$

Prop 9.3.9

Let $\mathcal{G}$ be a finite Cartan graph, X.

$\exists!$ morphism $w_0 \in \mathrm{Hom}(W(\mathcal{G}), X)$

s.t. $\ell(w) \le \ell(w_0)$ $\forall w \in \mathrm{Hom}(W(\mathcal{G}), X)$

Let $Y \in X$, $w_0 \in \mathrm{Hom}(Y, X)$

Then (1) $\Delta^{X\,\mathrm{re}}(w_0) = \Delta^{X\,\mathrm{re}}_+$

(2) w_0 is unique with the Property that $\forall i$,

$$\underline{\ell(w_0 s_i) < \ell(w_0)}$$

(3) $\forall\, \alpha \in \Delta^{\vee re}$, α is simple $\iff -w_0(\alpha)$ is simple.

Proof:

Since $\mathfrak{g}$ is finite,

by $\boxed{\text{Thm } 9.1.22}$ with $P_X = \Delta^{Xre}_+$

then $\exists\, w' \in \mathrm{Hom}(W(\mathfrak{g}), X)$

s.t. $\boxed{\Delta^{Xre}(w') = \Delta^{Xre}_+}$

Let $w \in \mathrm{Hom}(W(\mathfrak{g}), X)$ with $\underline{\ell(w) \geq \ell(w')}$

Then by Thm 9.3.5

$$\underline{|\Delta^{Xre}(w)|} = \ell(w) \geq |\Delta^{Xre}_+|$$

$$\Rightarrow \underline{\Delta^{Xre}(w)} = \Delta^{Xre}_+ = \underline{\Delta^{Xre}(w')}$$

$$\Rightarrow w = w'$$

This proves $w_0\ \exists!$ and (1).

For (2), ~~obvious~~.

w_0 has this property is obvious.

Conversely, let $w \in \mathrm{Hom}(W(\mathfrak{g}), X)$ has this property.

then $\underline{w(\alpha_i) \in -N_0 I}$, $\forall i$, by ...

~~then~~ Thus $\underline{\Delta^{Xre}(w) = \Delta^{Xre}_+}$

$$\Rightarrow w = w_0$$

pf: $w(\alpha_i) \in -\mathbb{N}_0^I \ \forall i$

$\Rightarrow$ by def $\underline{\Delta^{\boxed{Y}re}(w^{-1}) = \Delta^{Yre}_+}$

but $|\Delta^{Xre}(w)| = |\Delta^{Yre}(w^{-1})|$

$(N(w) = N(w^{-1}))$

$= |\Delta^{Xre}_+| \overset{\text{finite}}{=} |\Delta^{Yre}_+|$

(3)

Since $\underline{w_0(\alpha_i) \in -\mathbb{N}_0^I}$ by (1)

by (2)

(3) follows from $\underline{\text{Lem } 9.1.23}$.

$\therefore$ α is simple $\Leftrightarrow$ $-w_0(\alpha_i)$ is simple.

Def 9.3.10

Let $\mathcal{G}$ be a finite Cartan graph,

$X \in \mathcal{X}$,

The elmt w_0 in Prop 9.3.9 is called

$\underline{\text{the}}$ longest elmt in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$.

Cor 9.3.11

let $\mathcal{G}$ be a finite Cartan graph,

$X, Z \in \mathcal{X}$, $w \in \mathrm{Hom}(Z, X)$

Then any reduced decomposition $\underline{(i_1, \cdots, i_\ell)}$

of w can be extended to a reduced

decomposition $(i_1, \cdots, i_\ell, \cdots, i_m)$ of

$\textcircled{w_0} \in \mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$

Proof: By Prop 9.3.9 $\ell(w) < \ell(w_0)$ if $w \neq w_0$

Thus the claim follows from Prop 9.3.9 (2).

$w = (i_1, \dots, i_\ell)$

□

Cor 9.3.12

Let $\mathcal{G}$ be a connected finite Cartan graph,

Then $\mathcal{X}$ is finite and $\mathrm{Hom}(Y, X)$ is finite for all $X, Y \in \mathcal{X}$.

Proof: By Cor 9.3.8(2)

$$\Delta^{X\,re}(w) = \Delta^{X\,re}(w') \Rightarrow w' = w$$

the map from $\underline{\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)}$ to the power set of $\Delta_+^{X\,re}$ is injective

$$w \mapsto \Delta^{X\,re}(w) \subseteq \Delta_+^{X\,re}$$

Since $|\Delta_+^{X\,re}| < \infty$, we have $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is finite.

This proves the claim since $\mathcal{G}$ is connected.

Cor 9.3.13

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ finite Cartan graph, $w_0 = \mathrm{id}_X s_{i_1} \cdots s_{i_n}$ is the longest in $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$, and $\ell(w_0) = n$,

Then $\underline{n = |\Delta_+^{X\,re}|}$, $\Delta_+^{X\,re} = \Lambda^{X,\cancel{w_0}}_{(i_1, \dots, i_n)}$

$$\underline{\left\{ id_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \,\middle|\, 1 \le k \le n \right\}}$$

$\square$

Prop 9.3.14

Let $\mathcal{G}$ be a Cartan graph, $w \in W(\mathcal{G})$, $\underline{(i_1, \cdots, i_\ell)}$ with $\ell = \ell(w)$ a reduced decomposition of w.

For any $1 \le k \le \ell$, let $\beta_k = id_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$

Then $\Delta^{Xre}(w) = \{\beta_1, \cdots, \beta_\ell\}$ is totally ordered by $\beta_p < \beta_q \Leftrightarrow p < q$

Let $k, k_1, \cdots, k_r \in \{1, \cdots, \ell\}$, $k_1 \le k_2 \le \cdots \le k_r$

Assume $\beta_k = \sum_{i=1}^{r} \beta_{k_i}$,

Then either $r=1$, $k=k_1$

or $k_1 < k < k_r$.

Proof: Let $\underline{v = id_X s_{i_1} \cdots s_{i_{k-1}}}$

Then $\Delta^{Xre}(w) = \underline{\{\beta_1, \cdots, \beta_\ell\}}$

$\boxed{\Delta^{Xre}(v)} = \underline{\{\beta_1, \cdots, \beta_{k-1}\}}$

Thus $\underline{v^{-1}(\beta_j) \in -\mathbb{N}_0^I}$ for $1 \le j \le \ell$

$\Leftrightarrow j < k$

Moreover, $\alpha_{i_k} = v^{-1}(\beta_k)$

$= \sum_{i=1}^{r} v^{-1}(\beta_{k_i})$,

Have either $r=1$, $k=k_1$

or $r \geq 2$, $\boxed{k_1 < k < k_r}$.

$$\underbrace{1, \dots, k-1}_{<0},\ \underbrace{k}_{\alpha_{ik}},\ \underbrace{k+1, \dots, l}_{>0}$$

□

Prop 9.3.15 (关于 standard cartan graph)

Let $\mathcal{G} = \mathcal{G}(I, \mathcal{X}, r, A)$ be a $\downarrow$.

$X \in \mathcal{X}$.

(1) Let $Y \in \mathcal{X}$, $k, l \in \mathbb{N}_0$,

$i_1, \dots, i_k$, $j_1, \dots, j_l \in I$.

s.t. $\underline{\underline{\mathrm{id}_X}} s_{i_1} \cdots s_{i_k} = \underline{\underline{\mathrm{id}_X}} s_{j_1} \cdots s_{j_l}$

Then $\mathrm{id}_Y \underline{s_{i_1} \cdots s_{i_k}} = \mathrm{id}_Y \underline{s_{j_1} \cdots s_{j_l}}$.

Proof: By assumption,

$$F(\mathrm{id}_X s_{i_1} \cdots s_{i_k}) \boxed{=} F(\mathrm{id}_X s_{j_1} \cdots s_{j_k})$$

Since $A^Z = A^X$, $\forall Z \in \mathcal{X}$,

we have $$\underline{F(\underline{\underline{\mathrm{id}_Y}} s_{i_1} \cdots s_{i_k}) \boxed{=} F(\underline{\underline{\mathrm{id}_Y}} s_{j_1} \cdots s_{j_k})}$$

☆ $\boxed{\Rightarrow}$ $\underline{\mathrm{id}_Y s_{i_1} \cdots s_{i_k} = \mathrm{id}_Y s_{j_1} \cdots s_{j_k}}$

(2) The set $\mathrm{Hom}(\mathcal{W}(\mathcal{G}), X)$ is a group with "$\circ$".

$$\underline{(\mathrm{id}_X s_{i_1} \cdots s_{i_k})}\,\underline{(\mathrm{id}_X s_{j_1} \cdots s_{j_l})}$$

$$:= \underline{\mathrm{id}_X s_{i_1} \cdots s_{i_k} s_{j_1} \cdots s_{j_l}}.$$

"$\cdot$" is well~~ed~~-defined by (1)

$$(\mathrm{id}_X s_{i_1} \cdots s_{i_k})^{-1} = \mathrm{id}_X s_{i_k} \cdots s_{i_1}$$

(3) The group $\boxed{\mathrm{Hom}(W(\mathfrak{g}), X)}$ in (2) ~~[illegible]~~ $\cong$ the Weyl group of the Cartan matrix of $\mathfrak{g}$.

Proof:

The group $\mathrm{Hom}(W(\mathfrak{g}), X)$ is generated by $\boxed{\mathrm{id}_X s_i}$, $i \in I$.

and fulfills the relations

$$\boxed{\left((\mathrm{id}_X s_i)(\mathrm{id}_X s_j)\right)^{\boxed{m^X_{ij}}} = \mathrm{id}_X}$$

Since $\underline{\mathrm{id}_X (s_i s_j)^{m^X_{ij}} = \mathrm{id}_X}$

Thus $\exists!$ surj. group hom —

$$W \longrightarrow \mathrm{Hom}(W(\mathfrak{g}), X)$$

$$s_i \longmapsto \mathrm{id}_X s_i$$

injective: let $s_{i_1} \cdots s_{i_k} \in \ker$

i.e $\mathrm{id}_X s_{i_1} \cdots s_{i_k} = \mathrm{id}_X$

$\Rightarrow F(\mathrm{id}_X s_{i_1} \cdots s_{i_k}) = F(\mathrm{id}_X)$

$\Rightarrow s_{i_1} \cdots s_{i_k} = \mathrm{id}_{\mathbb{Z}I}$

$\square$